Journal of Sound and Vibration (2002) **249**(1), 189–195
doi:10.1006/jsvi.2000.3493, available online at http://www.idealibrary.com on IDEAL®

LETTERS TO THE EDITOR

MEMBRANE-LIKE VIBRATION OF SIMPLY SUPPORTED SPHERICAL SHALLOW SHELLS OF POLYGONAL PLANFORM

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(*Received* 17 *October* 2000)

1. INTRODUCTION

Exact correspondences between buckling and vibration eigenvalues of membranes and plates have been of renewed interest, recently. The establishment of such results enables one to obtain plate eigenvalues in terms of available membrane results. Such linking relationships have been found for single-layer homogeneous plates [1–5], sandwich plates [6–8], laminated plates [9] and functionally graded plates [8, 10] using different plate theories. However, these existing results are only limited to buckling and vibration of plates.

The present work further develops the links between vibration eigenvalues predicted by different theories. These correspondences are extended from a flat plate to a simply supported spherical shallow shell on a Winkler–Pasternak elastic foundation. The material of the shell is homogeneous and transversely isotropic. In analogy with a vibrating flat membrane, exact vibration frequencies of a homogeneous spherical shallow shell of polygonal planform are found using the classical theory and the first order and third order shear deformation theories. Some available results for single-layer homogeneous plates can be retrieved from the present results.

2. GOVERNING EQUATIONS

Consider a homogeneous spherical shallow shell of uniform thickness h, whose projected planform is polygonal. The rectangular Cartesian planform co-ordinates x_1 and x_2 are introduced in the deformation analysis of the shallow shell. The reference surface is the middle surface of the shell defined by $x_3 = 0$, and x_3 denotes the thickness co-ordinate measured from the undeformed middle surface. Hereafter, a comma followed by a subscript i denotes the partial derivative with respect to x_i, and a repeated index implies summation over the range of the index with Latin indices ranging from 1 to 3 and Greek indices from 1 to 2.

The following third order displacement field is assumed for the shallow shell:

$$v_\alpha(x_i; t) = u_\alpha - x_3 u_{3,\alpha} + g\varphi_\alpha, \qquad v_3(x_i; t) = u_3, \tag{1}$$

where u_α, u_3 and φ_α are independent of x_3, and

$$g\,(x_3) = x_3\left(1 - \frac{4x_3^2}{3h^2}\right). \tag{2}$$

0022-460X/02/010189 + 07 $35.00/0

The displacement field (1) is essentially the same as that of Reddy's third order theory for composite laminated plates [11]. It can also be seen that by taking $g(x_3) = x_3$ and $g(x_3) = 0$, the displacement field (1) will become that of the first order shear deformation theory and of the classical theory respectively [12].

Denoting k and G as the Winkler–Pasternak elastic foundation parameters [13], r the middle surface radius of the spherical shallow shell, ρ the mass density, and ω an angular frequency, the time-harmonic linear governing equations for the spherical shallow shell resting on a Winkler–Pasternak elastic foundation are expressed as

$$N_{\alpha\beta,\beta} + \rho i_0 \omega^2 u_\alpha = 0, \tag{3}$$

$$M_{\alpha\beta,\alpha\beta} - \frac{N_{\alpha\alpha}}{r} - ku_3 + Gu_{3,\alpha\alpha} + \rho i_0 \omega^2 u_3 - \rho i_1 \omega^2 u_{3,\alpha\alpha} + \rho i_2 \omega^2 \varphi_{\alpha,\alpha} = 0, \tag{4}$$

$$P_{\alpha\beta,\beta} - R_\alpha - \rho i_2 \omega^2 u_{3,\alpha} + \rho i_3 \omega^2 \varphi_\alpha = 0, \tag{5}$$

where

$$[i_0, i_1, i_2, i_3] = \int_{-h/2}^{h/2} [1, x_3^2, x_3 g, g^2]\, \mathrm{d}x_3, \tag{6}$$

$$[N_{\alpha\beta}, M_{\alpha\beta}, P_{\alpha\beta}] = \int_{-h/2}^{h/2} \sigma_{\alpha\beta}\, [1, x_3, g]\, \mathrm{d}x_3, \qquad R_\alpha = \int_{-h/2}^{h/2} \sigma_{\alpha 3} g_{,3}\, \mathrm{d}x_3, \tag{7}$$

$$\sigma_{\alpha\beta} = H_{\alpha\beta\omega\rho}\, e_{\omega\rho}, \qquad \sigma_{\alpha 3} = 2E_{\alpha 3\omega 3}\, e_{\omega 3}, \tag{8}$$

$$e_{\alpha\beta} = \frac{1}{2}(v_{\alpha,\beta} + v_{\beta,\alpha}) + \frac{v_3}{r}\,\delta_{\alpha\beta}, \qquad e_{\alpha 3} = \frac{1}{2}(v_{\alpha,3} + v_{3,\alpha}). \tag{9}$$

In equations (3)–(5) and thereafter, the time-harmonic factor $\exp(\mathrm{i}\omega t)$ has been omitted and each physical quantity refers to its spatial part. For the classical theory and the first order theory to be included, the integrals in equation (6) are not explicitly performed. Specifically, $g(x_3) = x_3$ leads to $i_1 = i_2 = i_3$ for the first order theory and $g(x_3) = 0$ leads to $i_2 = i_3 = 0$ for the classical theory.

The material properties for a transversely isotropic material are

$$H_{\alpha\beta\omega\rho} = \frac{vE}{1 - v^2}\,\delta_{\alpha\beta}\delta_{\omega\rho} + \frac{E}{2(1 + v)}(\delta_{\alpha\omega}\delta_{\beta\rho} + \delta_{\alpha\rho}\delta_{\beta\omega}), \qquad E_{\alpha 3\omega 3} = \mu\delta_{\alpha\omega}, \tag{10}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta, E and v are Young's modulus and the Poisson ratio in the surface of isotropy, and μ is the shear modulus normal to the isotropy surface. In particular, $\mu = E/2(1 + v)$ for an isotropic material.

Equations (7) may be written alternatively in terms of displacements as

$$N_{\alpha\beta} = (a - b) i_0 u_{\omega,\omega}\delta_{\alpha\beta} + \frac{b i_0}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) + \frac{(2a - b) i_0}{r} u_3 \delta_{\alpha\beta},$$

$$M_{\alpha\beta} = (a - b)(-i_1 u_{3,\omega\omega} + i_2 \varphi_{\omega,\omega})\delta_{\alpha\beta} - b i_1 u_{3,\alpha\beta} + \frac{b i_2}{2}(\varphi_{\alpha,\beta} + \varphi_{\beta,\alpha}),$$

$$P_{\alpha\beta} = (a - b)(-i_2 u_{3,\omega\omega} + i_3 \varphi_{\omega,\omega})\delta_{\alpha\beta} - b i_2 u_{3,\alpha\beta} + \frac{b i_3}{2}(\varphi_{\alpha,\beta} + \varphi_{\beta,\alpha}), \qquad R_\alpha = c\varphi_\alpha, \tag{11}$$

where

$$a = \frac{E}{1 - \nu^2}, \qquad b = \frac{E}{1 + \nu}, \qquad c = \mu \int_{-h/2}^{h/2} (g_{,3})^2 \, \mathrm{d}x_3. \tag{12}$$

With equations (11), the governing equations (3)–(5) are expressed in terms of five displacement functions u_α, u_3 and φ_α as

$$\frac{bi_0}{2} u_{\alpha,\beta\beta} + \frac{(2a-b)i_0}{2} u_{\beta,\beta\alpha} + \frac{(2a-b)i_0}{r} u_{3,\alpha} + \rho i_0 \omega^2 u_\alpha = 0, \tag{13}$$

$$- ai_1 u_{3,\alpha\alpha\beta\beta} + ai_2 \varphi_{\alpha,\alpha\beta\beta} - \frac{(2a-b)i_0}{r} u_{\alpha,\alpha} - \frac{2(2a-b)i_0}{r^2} u_3 - ku_3 + Gu_{3,\alpha\alpha}$$

$$+ \rho i_0 \omega^2 u_3 - \rho i_1 \omega^2 u_{3,\alpha\alpha} + \rho i_2 \omega^2 \varphi_{\alpha,\alpha} = 0, \tag{14}$$

$$- ai_2 u_{3,\alpha\beta\beta} + \frac{bi_3}{2} \varphi_{\alpha,\beta\beta} + \frac{(2a-b)i_3}{2} \varphi_{\beta,\beta\alpha} - c\varphi_\alpha - \rho i_2 \omega^2 u_{3,\alpha} + \rho i_3 \omega^2 \varphi_\alpha = 0. \tag{15}$$

The following matrix equation can be obtained through equation (14) and differentiating equations (13) and (15) with respect to x_α,

$$\mathbf{KX} = \mathbf{0}, \tag{16}$$

where

$$\mathbf{X} = [u_{\alpha,\alpha} \; u_3 \; \varphi_{\alpha,\alpha}]^{\mathrm{T}} \qquad \mathbf{0} = [0 \; 0 \; 0]^{\mathrm{T}}, \tag{17}$$

and $\mathbf{K} = (K_{IJ})$ is a 3×3 operator matrix in which its elements, expressed in terms of the two-dimensional Laplace operator ∇^2, are

$$K_{11}(\nabla^2) = ai_0 \nabla^2 + \rho i_0 \omega^2, \qquad K_{12}(\nabla^2) = \frac{(2a-b)i_0}{r} \nabla^2, \qquad K_{13}(\nabla^2) = 0,$$

$$K_{21}(\nabla^2) = -\frac{(2a-b)i_0}{r}, \qquad K_{22}(\nabla^2) = - ai_1 \nabla^4 + (G - \rho i_1 \omega^2) \nabla^2$$

$$- \frac{2(2a-b)i_0}{r^2} - k + \rho i_0 \omega^2,$$

$$K_{23}(\nabla^2) = ai_2 \nabla^2 + \rho i_2 \omega^2, \qquad K_{31}(\nabla^2) = 0,$$

$$K_{32}(\nabla^2) = - ai_2 \nabla^4 - \rho i_2 \omega^2 \nabla^2, \qquad K_{33}(\nabla^2) = ai_3 \nabla^2 - c + \rho i_3 \omega^2. \tag{18}$$

Furthermore, eliminating $u_{\alpha,\alpha}$ and $\varphi_{\alpha,\alpha}$ from equation (16) gives

$$\det [\mathbf{K}(\nabla^2)] u_3 = a^3 i_0 (i_2^2 - i_1 i_3)(\nabla^2 + \lambda_1)(\nabla^2 + \lambda_2)(\nabla^2 + \lambda_3)(\nabla^2 + \lambda_4) u_3 = 0, \tag{19}$$

where λ_1, λ_2, λ_3 and λ_4 are four roots of the quartic equation

$$\det[\mathbf{K}(-\lambda)] = 0. \tag{20}$$

Equation (19) is the characteristic equation from which the eigenvalues and associated eigenfunctions for vibration problems of the homogeneous spherical shallow shell can be solved under given boundary conditions.

3. SIMPLY SUPPORTED EDGES OF A SPHERICAL SHALLOW SHELL OF POLYGONAL PLANFORM

Assume that the homogeneous spherical shallow shell is simply supported on its edges for which the boundary conditions are

$$u_3 = 0, \quad u_T = 0, \quad \varphi_T = 0, \quad u_{3,T} = 0, \quad N_{NN} = 0, \quad M_{NN} = 0, \quad P_{NN} = 0, \tag{21}$$

where the upper case subscripts N and T denote, respectively, normal and tangential directions to the boundary, and the summation convention does not apply to them. For a spherical shallow shell of polygonal planform, equation $(21)_4$ is identically satisfied due to equation $(21)_1$. In terms of equations $(11)_{1\text{-}3}$ and $(21)_{1\text{-}3}$, the boundary conditions $(21)_{5\text{-}7}$ reduce to

$$u_{N,N} = 0, \quad u_{3,NN} = 0, \quad \varphi_{N,N} = 0. \tag{22}$$

Therefore, the boundary conditions for the simply supported spherical shallow shell of polygonal planform can be expressed as

$$u_T = 0, \quad u_{\alpha,\alpha} = 0, \quad u_3 = 0, \quad \nabla^2 u_3 = 0, \quad \varphi_T = 0, \quad \varphi_{\alpha,\alpha} = 0, \tag{23}$$

and furthermore, by using equations (16),

$$\nabla^{2J} u_{\alpha,\alpha} = 0, \quad \nabla^{2J} u_3 = 0, \quad \nabla^{2J} \varphi_{\alpha,\alpha} = 0 \quad (J = 0, 1, 2, \ldots). \tag{24}$$

4. MEMBRANE ANALOGY

In order to facilitate subsequent analysis, equation (19) is written as

$$(\nabla^2 + \lambda_1)H_1 = 0, \quad H_1 \equiv a^3 i_0 (i_2^2 - i_1 i_3)(\nabla^2 + \lambda_2)(\nabla^2 + \lambda_3)(\nabla^2 + \lambda_4)u_3, \tag{25}$$

where λ_1 can be any one of the four roots of the quartic equation (20). In view of equations $(24)_2$ and $(25)_2$, the Helmholtz equation $(25)_1$ is shown to be associated with the following boundary condition:

$$H_1 = 0. \tag{26}$$

Therefore, the eigenvalue problem for the third order theory of a homogeneous spherical shallow shell of polygonal planform reduces to Dirichlet's boundary value problem, equation $(25)_1$ and the boundary condition (26). This boundary value problem is mathematically similar to a uniform membrane whose shape coincides with the same contour as the shell planform, is fixed at the edges and executes small transverse vibration.

Thus, the eigenvalue λ_1 may be designated as that of a vibrating membrane with the same contour as the shell planform, i.e.

$$\lambda_1 = \lambda_M, \tag{27}$$

where $\lambda_M = \rho_M \omega_M^2 / Y$ is the eigenvalue of the membrane vibration problem [14], ρ_M, Y and ω_M are the mass density, constant tension and vibration frequency of the membrane respectively. Because λ_1 is a root of the quadratic equation (20), substituting equation (27) into equation (20) yields

$$\det[\mathbf{K}(-\lambda_M)] = 0 \tag{28}$$

or, after some rearrangements,

$$\det(-\lambda_M \mathbf{R} + \rho\omega^2 \mathbf{S}) = 0, \tag{29}$$

where

$$\mathbf{R} = \begin{bmatrix} a i_0 & \dfrac{(2a-b)i_0}{r\lambda_M} & 0 \\ \dfrac{(2a-b)i_0}{r\lambda_M} & a i_1 + \dfrac{2(2a-b)i_0}{r^2\lambda_M^2} + \dfrac{G}{\lambda_M} + \dfrac{k}{\lambda_M^2} & a i_2 \\ 0 & a i_2 & a i_3 + \dfrac{c}{\lambda_M} \end{bmatrix}, \qquad \mathbf{S} = \begin{bmatrix} i_0 & 0 & 0 \\ 0 & i_1 + \dfrac{i_0}{\lambda_M} & i_2 \\ 0 & i_2 & i_3 \end{bmatrix}. \tag{30}$$

For the free-vibration problem using the third order theory for the homogeneous spherical shallow shell of polygonal planform, the eigenfrequencies can be simply obtained from equation (29), which is a cubic equation with respect to ω^2 and thus gives three eigenfrequencies. The eigenvectors associated with these vibration frequencies exhibit flexural and thickness-shear modes, as well as a stretching mode. The spherical shallow shell will execute a motion in coupled form of the stretching, flexural and thickness-shear modes.

5. THE FIRST ORDER THEORY

When taking $g(x_3) = x_3$, it can be seen from equation (1) that the displacement field is essentially the one for the first order theory. It follows from equation (6) that

$$i_1 = i_2 = i_3 = h^3/12. \tag{31}$$

In addition, it is conventional to introduce the shear correction factor κ in the first order theory, i.e. the parameter c defined by equation $(12)_3$ should be replaced by

$$c_F = \kappa\mu h. \tag{32}$$

The characteristic equation using the first order theory for the homogeneous spherical shallow shell of polygonal planform is the same as the matrix equation (16), which upon eliminating $u_{\alpha,\alpha}$ and $\varphi_{\alpha,\alpha}$ reduces to a cubic equation with respect to the Laplace operator ∇^2, i.e. a degenerated form of equation (19) due to equation (31).

Although the boundary conditions for simply supported edges are slightly different from those in equations (24), it can be shown by following the same procedure as in section 4 that the free-vibration frequencies of the homogeneous spherical shallow shell of polygonal planform are given by the same equation (29) wherein the relations (31) should be incorporated and c should be replaced by c_F.

6. THE CLASSICAL THEORY

Equation (1) with $g(x_3) = 0$ precisely represents the displacement field of the classical Kirchhoff theory for the homogeneous spherical shallow shell. In this case,

$$i_2 = i_3 = 0, \qquad c = 0. \tag{33}$$

Similarly, the free vibration frequency ω_K^2 can be solved from

$$\det\left\{ -\lambda_M \begin{bmatrix} a i_0 & \dfrac{(2a-b)i_0}{r\lambda_M} \\ \dfrac{(2a-b)i_0}{r\lambda_M} & a i_1 + \dfrac{2(2a-b)i_0}{r^2\lambda_M^2} + \dfrac{G}{\lambda_M} + \dfrac{k}{\lambda_M^2} \end{bmatrix} + \rho\omega_K^2 \begin{bmatrix} i_0 & 0 \\ 0 & i_1 + \dfrac{i_0}{\lambda_M} \end{bmatrix} \right\} = 0. \tag{34}$$

Note that only two free vibration frequencies are predicted by the classical theory. This is because of the assumption that a normal to the mid-surface of the shell remains normal to mid-surface during deformation. Consequently, there is no thickness-shear motion in the classical theory.

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